# Physics of Fluids

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# Chapter 1. Introduction

## 1.1 Fluid Mechanics

**Fluid Mechanics** is the branch of science concerned with the behaviour of liquids and gases at rest and in motion.

This is very important, since fluids are present everywhere, in nature, biology and engineering.

The range of length scales and flow speeds is enormous, but interestingly, they are all governed by the same equations.

## 1.2 Units of Measurement

For mechanical systems, the units of all physical variables can be expressed in terms of the units of four basic variables.

| **Quantity** | **Name of unit** | **Symbol** | **Equivalent** |
|---|---|---|---|
| Length | Meter | m | |
| Mass | Kilogram | kg | |
| Time | Second | s | |
| Temperature | Kelvin | K | |
| Frequency | Hertz | Hz | $s^{-1}$ |
| Force | Newton | N | $kg\ m\ s^{-2}$ |
| Pressure | Pascal | Pa | $N\ m^{-2}$ |
| Energy | Joule | J | N m |
| Power | Watt | W | $J\ s^{-1}$ |

## 1.3 Solids, Liquids, and Gases

### 1.3.1 Fluids

A **fluid** (i.e. liquids or gases) deforms continuously when *however small* shear stress is applied. It does not have a preferred shape. Fluids generally fall into two classes, *liquids and gases*. A gas always expands to fill the entire volume of its container. In contrast, the volume of a liquid changes little, so that it may not completely fill a large container.

### 1.3.2 Solids

A **solid** does not deform continuously when a shear stress is applied and it relaxes back to a preferred shape when this stress is removed. An elastic solid has a perfect memory of its preferred shape (because it always springs back to its preferred shape when unloaded) and an ordinary viscous fluid has zero memory (because it never springs back when unloaded). Some substances can be called **viscoelastic** because they partially rebound when unloaded.

### 1.3.3 Stresses

Solids and fluids behave very differently when subjected to *shear stresses*, they behave similarly under the action of *compressive normal stresses*. However, *tensile normal stresses* again lead to differences in fluid and solid behaviour.

### 1.3.4 Newtonian Fluid

A **Newtonian fluid** is a fluid in which the viscous stresses arising from its flow are at every point linearly correlated to the local strain rate, the rate of change of its deformation over time. Newtonian fluids are the simplest mathematical models of fluids that account for viscosity.

Newton's law of friction is given by:

$$\tau = \mu \frac{du}{dy} \tag{1.1}$$

$\tau = F/A$ is the shear stress, with $A$ the area of the plate and $\mu$ is the viscosity.

## 1.4 Continuum Hypothesis

A fluid is discontinuous or discrete at the most microscopic scales. The *average manifestation* of molecular motions is more important for macroscopic fluid mechanics.
When the molecular density of the fluid and the size of the region of interest are large enough, such average properties are sufficient for explaining macroscopic phenomena and the discrete molecular structure of matter may be ignored and replaced with a continuous distribution, called a **continuum**.

**Continuum assumption:** in a continuum, fluid properties like density or velocity are defined to be appropriate averages of molecular characteristics in a small region $\delta V$ surrounding the point of interest.

## 1.6 Surface Tension

**Surface tension** $\sigma$ is defined as the magnitude of the tensile force that acts per unit length to open a line segment lying on the surface. It has dimension N m$^{-1}$.
Alternatively, $\sigma$ can be thought of as the energy needed to create a unit of interfacial area. In general, $\sigma$ depends on the pair of fluids in contact, the temperature, and the presence of surface-active chemicals (surfactants) or impurities, even at very low concentrations.

Force equilibrium for spherical droplet yields:

$$\sigma \cdot (2\pi R) = (p_{\text{inside}} - p_{\text{outside}})\pi R^2 \tag{1.2}$$

or

$$p_i - p_o = 2\sigma/R \tag{1.3}$$

$p_i - p_o$ is the **Laplace pressure**.

If one puts a droplet on a surface, its shape depends on the 3 different materials (surface (solid), droplet (liquid) and surrounding (gas)), or the surface tensions of the 3 different materials. This shape can be characterized by the contact angle.

Horizontal equilibrium yields **Young's equations**:

$$\sigma_{\text{SG}} = \sigma_{\text{LG}} \cos(\theta_{\text{C}}) + \sigma_{\text{SL}} \tag{1.4}$$

From which the definition of the **contact angle** follows:

$$\cos(\theta_{\text{C}}) = \frac{\sigma_{\text{SG}} - \sigma_{\text{SL}}}{\sigma_{\text{LG}}} \tag{1.5}$$

| | | contact angle |
|---|---|---|
| $\sigma_{\text{SG}} < \sigma_{\text{SL}}$ | hydrophobic | $105^o - 120^o$ |
| $\sigma_{\text{SG}} > \sigma_{\text{SL}}$ | hydrophilic | $0^o - 30^o$ |

The free surface of a liquid in a narrow tube (*capillary tubes*) rises above the surrounding level due to the influence of surface tension.

## 1.7 Fluid Statics

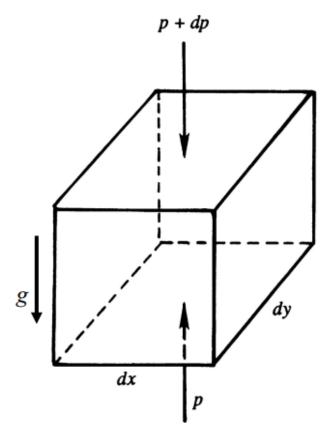
The magnitude of the force per unit area in a static fluid is called the **pressure**. Sometimes the ordinary pressure is called the **absolute pressure**, in order to distinguish it from the **gauge pressure**, which is defined as the absolute pressure minus the local atmospheric pressure:

$$p_{\text{gauge}} = p - p_{\text{atm}} \tag{1.6}$$

$p$ is the absolute pressure, $p_{\text{atm}}$ is 101.3 kPa = 1.013 Bar.

In a fluid at rest, tangential shear stresses are absent and the only force between adjacent surfaces is normal to the surface. As a result, the pressure is equal in all directions. Shear stresses are absent, otherwise, the fluid wouldn't be at rest (see Newton's law of friction).

The spatial distribution of pressure in a static fluid can be determined from a three-dimensional force balance. Consider an infinitesimal cube of sides $dx$, $dy$, and $dz$, with the $z$-axis vertically upward.




It is clear that:

$$\partial p/\partial x = \partial p/\partial y = 0$$

The balance of forces is expressed by **Pascal's law**, which states that all points in a resting fluid medium (and connected by the same fluid) are at the same pressure if they are at the same depth.

$$\frac{dp}{dz} = -\rho g \tag{1.7}$$

Integration (assuming constant density) gives:

$$p = p_0 - \rho g z \tag{1.8}$$

Where $p_0$ is the pressure at $z = 0$.

We can obtain an approximate expression for the pressure distribution of the atmosphere by combining this with the perfect gas law $p = \rho RT$ at isothermal conditions ($T = 250$ K):

$$dp/dz = -\rho g = -pg/RT \longrightarrow p(z) = p_0 e^{-gz/RT} \tag{1.9}$$

The quantity $RT/g = 7.3$ km is called the scale height of the atmosphere and gives an estimate of its thickness.

## 1.11 Dimensional Analysis

From the problem geometry, boundary conditions, initial conditions, material parameters and one solution variable, find the $n$ dimensional (system) parameters $q_1$, $q_2$, $q_3$, $\ldots$, $q_n$. Buckingham's theorem states that these n variables can be combined into $n-r$ dimensionless groups $\Pi_i(i=1,n-r)$, so that the solution of the problem can be written as

$$\Pi_1 = \varphi\left(\Pi_2, \Pi_3, \ldots, \Pi_{n-r}\right)$$

Here, $r$ is the number of independent dimensions and $\Pi_1$ includes the solution variable.

*Now, how the fuck do you do this?*

#### Step 1. Select Variables and Parameters

Create a list of variables and parameters that appear in the problem.

#### Step 2. Create the Dimensional Matrix

Express the dimensions of all the variables in terms of four basic dimensions: mass $M$, length $L$, time $T$, and temperature $\theta$.

#### Step 3. Determine the Rank of the Dimensional Matrix

The rank, $r$ of this matrix is given by the number of basic dimensions that occur in the problem.

#### Step 4. Determine the Number of Dimensionless Groups

The number of dimensionless groups is $n-r$ where $n$ is the number of variables and parameters, and $r$ is the rank of the dimensional matrix.

#### Step 5. Construct the Dimensionless Groups

Select $r$ parameters from the dimensional matrix as repeating parameters that will be found in all the subsequently constructed dimensionless groups. These repeating parameters must span the appropriate $r$-dimensional space of $M$, $L$, and/or $T$. For many fluid-flow problems, a characteristic velocity, a characteristic length, and a fluid property involving mass are ideal repeating parameters.

#### Step 6. State the Dimensionless Relationship

This step merely involves placing the $(n-r)\Pi-$groups in the right form.

$$\phi\left(\Pi_1, \Pi_2, \ldots, \Pi_{n-r,}\right) = 0 \quad \text{or} \quad \Pi_1 = \varphi\left(\Pi_2, \Pi_{3,}, \ldots, \Pi_{n-r,}\right)$$

#### Step 7. Use Physical Reasoning or Additional Knowledge to Simplify the Dimensionless Relationship

Sometimes there are only two extensive thermodynamic variables involved and these must be proportional in the final scaling law. An overall conservation law can be applied that restricts one or more parametric dependencies or a phenomenon may be known to be linear, quadratic, etc. in one of the parameters and this dependence must be reflected in the final scaling law.

### 1.11.1 Commonly Used Parameters and Their Units

| Quantity | Symbol | M | L | T |
|---|---|---|---|---|
| Velocity | $U$ | 0 | 1 | -1 |
| Density | $\rho$ | 1 | -3 | 0 |
| Viscosity | $\mu$ | 1 | -1 | -1 |
| Surface tension | $\sigma$ | 1 | 0 | -2 |
| Friction | $\tau$ | 1 | -1 | -2 |
| Gravity | $g$ | 0 | 1 | -2 |
| Frequency | $f$ | 0 | 0 | -1 |
| Force | $F$ | 1 | 1 | -2 |
| Pressure | $p$ | 1 | -1 | -2 |
| Energy | $E$ | 1 | 2 | -2 |
| Power | $P$ | 1 | 2 | -3 |

# Chapter 2. Cartesian Tensors

## 2.1 Scalars, Vectors, Tensors, Notation

### 2.1.1 Zero-Order Tensors

**Scalars** or **zero-order tensors** may be defined with a single magnitude and appropriate units, and may vary with spatial location, but are independent of coordinate directions.

### 2.1.2 First-Order Tensors

**Vectors** or **first-order tensors** have both a magnitude and a direction. A vector can be completely described by its components along three orthogonal coordinate directions. Thus, the components of a vector may change with a change in coordinate system. In a Cartesian coordinate system with unit vectors $\mathbf{e_1}$, $\mathbf{e_2}$, and $\mathbf{e_3}$, in the three mutually perpendicular directions, the position vector $\mathbf{x}$ may be written:

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$

For algebraic manipulation, a vector is written as a column matrix:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ where } \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

### 2.1.3 Second-Order Tensors

**Second-order tensors** have a component for each pair of coordinate directions and therefore may have as many as $3 \times 3 = 9$ separate components.

### 2.1.4 Einstein Summation Convention

A second implicit feature of *index-based* or *indicial* notation is the implied sum over a repeated index in terms involving multiple indices. This notational convention can be stated as follows: *whenever an index is repeated in a term, a summation over this index is implied, even though no summation sign is explicitly written.* This is sometimes referred to as the **Einstein summation convention**.

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} = u_1v_1 + u_2v_2 + u_3v_3 = \sum_{i=1}^{3} u_iv_i \equiv u_iv_i$$

## 2.2 Rotation of Axes: Formal Definition of a Vector

Let $O123$ be the original coordinate system, and $O1'2'3'$ be the rotated system that shares the same origin O. The position vector $\mathbf{x}$ can be written in either coordinate system:

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3, \quad \text{or} \quad \mathbf{x} = x_1'\mathbf{e}_1' + x_2'\mathbf{e}_2' + x_3'\mathbf{e}_3'$$

Coordinate can be found by taking the inner product with axis $\mathbf{e}_1'$:

$$\mathbf{x} \cdot \mathbf{e}_1' = x_1\mathbf{e}_1 \cdot \mathbf{e}_1' + x_2\mathbf{e}_2 \cdot \mathbf{e}_1' + x_3\mathbf{e}_3 \cdot \mathbf{e}_1' = x_1'$$

where the dot products between unit vectors are direction cosines.

Forming the dot products $\mathbf{x} \cdot \mathbf{e}_2' = x_2'$ and $\mathbf{x} \cdot \mathbf{e}_3' = x_3'$, and then combining these results produces:

$$x_j' = x_1C_{1j} + x_2C_{2j} + x_3C_{3j} = \sum_{j=1}^{3} x_iC_{ij} \equiv x_iC_{ij}$$

where $C_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j'$ is a $3 \times 3$ matrix of direction cosines. Here, the free or not-summed-over index is $j$, while the repeated or summed-over index can be any letter other than $j$.

$$C_{ij} = \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{e}_r & \mathbf{e}_1 \cdot \mathbf{e}_\theta \\ \mathbf{e}_2 \cdot \mathbf{e}_r & \mathbf{e}_2 \cdot \mathbf{e}_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

## 2.3 Multiplication of Matrices

Let **A** and **B** be two $3 \times 3$ matrices. The inner product of **A** and **B** is defined as the matrix **P** whose elements are related to those of **A** and **B** by:

$$P_{ij} = \sum_{k=1}^{3} A_{ik} B_{kj} \equiv A_{ik} B_{kj}, \text{ or } \quad \mathbf{P} = \mathbf{A} \cdot \mathbf{B}$$

In explicit form, this is written as:

$$\begin{bmatrix} P_{11} & \boxed{P_{12}} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

The dashed box denotes the entry:

$$P_{12} = A_{11} B_{12} + A_{12} B_{22} + A_{13} B_{32}$$

## 2.4 Second-Order Tensors

Nearly all the tensors considered in Newtonian fluid mechanics are zero-, first-, or second-order tensors. A second-order tensor can be represented by nine components, one for each pair of directions and denoted by two free indices.

Consider the stress tensor $\tau$ or $\tau_{ij}$. Its two free indices specify two directions; the first $(i)$ index of $T_{ij}$ denotes the direction of the surface normal, and the second $(j)$ index denotes the force component direction.

The state of stress at a point can be completely specified by the nine components $\tau_{ij}$ (where $i$, $j = 1, 2, 3$) that can be written as the matrix:

$$\tau = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

It can be shown by a force balance on a tetrahedron element that the components of **T** in the rotated coordinate system are:

$$\tau'_{mn} = \sum_{i=1}^{3} \sum_{j=1}^{3} C_{im} C_{jn} \tau_{ij} \equiv C_{im} C_{jn} \tau_{ij}$$

For example, **A** is a fourth-order tensor if it has four free indices, and the associated $3^4 = 81$ components change under a rotation of the coordinate system according to:

$$A'_{m\nu q} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \sum_{l=1}^{3} C_{im} C_{jn} C_{kp} C_{lq} A_{ijkl} \equiv C_{im} C_{jn} C_{kp} C_{lq} A_{ijkl}$$

## 2.5 Contraction and Multiplication

When the two indices of a tensor are equated, and a summation is performed over this repeated index, the process is called *contraction*. An example is:

$$\sum_{j=1}^{3} A_{jj} \equiv A_{jj} = A_{11} + A_{22} + A_{33}$$

which forms a scalar.

When two second-order tensors **A** and **B** are multiplied a fourth-order tensor is formed with components $A_{ij}B_{kl}$. Lower-order tensors can be formed by performing a contraction ('dot-product' or inner product), leading to a second-order tensor:

$$\sum_{i=1}^{3} A_{ij}B_{jk} \equiv A_{ij}B_{jk} = (\mathbf{A} \cdot \mathbf{B})_{ik}$$

or, when a 2th-order tensor is multiplied in contracted form with a vector, a vector is formed:

$$\sum_{i=1}^{3} A_{ij}u_j \equiv A_{ij}u_j = (\mathbf{A} \cdot \mathbf{u})_i$$

and, finally, the double contraction, or 'double dot product' is defined as

$$\sum_{i=1}^{3}\sum_{j=1}^{3} A_{ij}B_{ji} \equiv A_{ij}B_{ji} (= \mathbf{A} : \mathbf{B})$$

resulting in a scalar.

## 2.6 Force on a Surface

The force on a surface per unit area, $\mathbf{f}(\mathbf{n})$ with components $f_i$ is given by:

$$f_i = \sum_{j=1}^{3} \tau_{ji}n_j \equiv \tau_{ji}n_j \quad \text{or} \quad \mathbf{f} = \mathbf{n} \cdot \tau$$

which is similar to finding the components of a vector along **n**:

$$u_n = \mathbf{u} \cdot \mathbf{n},$$

although $u_n$ is a scalar and **f** is a vector.

## 2.7 Kronecker Delta and Alternating Tensor

The Kronecker delta is defined as:

$$\delta_{ij} = \left\{ \begin{array}{l} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{array} \right\}$$

and has the following properties:

$$\delta_{ij}u_j = u_i, \qquad \delta_{ij}\delta_{jk} = \delta_{ik}, \qquad \text{and,} \qquad \delta_{ii} = 3$$

The alternating tensor or permutation symbol is defined as:

$$\varepsilon_{ijk} = \left\{ \begin{array}{l} \ \ 1 \text{ if } ijk = 123, 231, \text{ or } 312 \text{ (cyclic order)}, \\ \ \ 0 \text{ if any two indices are equal}, \\ -1 \text{ if } ijk = 321, 213, \text{ or } 132\text{( anti-cyclic order )} \end{array} \right\}$$

From this definition, it is clear that an index on $\varepsilon_{ijk}$ can be moved two places (either to the right or to the left) without changing its value.

A very frequently used relation is the *epsilon delta relation*:

$$\sum_{k=1}^{3} \varepsilon_{ijk}\varepsilon_{klm} \equiv \varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Additionally:

$$\varepsilon_{ijk}\varepsilon_{ijk} = 6$$

## 2.8 Vector Dot and Cross Products

Dot-product

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{i=1}^{3} u_i v_i \equiv u_i v_i$$

Cross-product

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\,\mathbf{e}_1 + (u_3 v_1 - u_1 v_3)\,\mathbf{e}_2 + (u_1 v_2 - u_2 v_1)\,\mathbf{e}_3$$

Alternatively, the cross product can be written as a determinant:

$$\mathbf{u} \times \mathbf{v} = \det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

or, in indicial notation,

$$(\mathbf{u} \times \mathbf{v})_k = \sum_{i=1}^{3} \sum_{j=1}^{3} \varepsilon_{ijk} u_i v_j \equiv \varepsilon_{ijk} u_i v_j = \varepsilon_{kij} u_i v_j$$

## 2.9 Gradient, Divergence, and Curl

Del-operator

$$\nabla = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3} = \sum_{i=1}^{3} \mathbf{e}_i \frac{\partial}{\partial x_i} \equiv \mathbf{e}_i \frac{\partial}{\partial x_i}$$

Gradient of a scalar function

$$\nabla \phi = \sum_{i=1}^{3} \mathbf{e}_i \frac{\partial \phi}{\partial x_i} \equiv \mathbf{e}_i \frac{\partial \phi}{\partial x_i}$$

The gradient of $\phi$ is perpendicular to surfaces defined by $\phi =$ constant. The spatial rate of change of $\phi$ in the direction of $\mathbf{n}$ is defined as

$$\partial \phi / \partial n = \nabla \phi \cdot \mathbf{n}$$

Divergence of a vector field

$$\nabla \cdot \mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} \equiv \frac{\partial u_i}{\partial x_i}$$

Divergence of a second order tensor (e.g. stress)

$$(\nabla \cdot \tau)_i = \sum_{j=1}^{3} \frac{\partial \tau_{ij}}{\partial x_j} \equiv \frac{\partial \tau_{ij}}{\partial x_j}$$

curl of a vector field

$$(\nabla \times \mathbf{u})_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} \equiv \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j}$$

A vector-field is divergence-free (solenoidal) when $\nabla \cdot \mathbf{u} = 0$, and curl-free (irrotational) when $\nabla \times \mathbf{u} = 0$.

## 2.10 Symmetric and Antisymmetric Tensors

A tensor $\mathbf{B}$ is symmetric when $B_{ij} = B_{ji}$ and anti-symmetric when $B_{ij} = -B_{ji}$. Any tensor can be represented by the sum of a symmetric $S_{ij}$ and antisymmetric tensor $A_{ij}$:

$$B_{ij} = \frac{1}{2}\left(B_{ij} + B_{ji}\right) + \frac{1}{2}\left(B_{ij} - B_{ji}\right) = S_{ij} + A_{ij}$$

As a result, the double contraction of a symmetric tensor $T_{ij}$ and another tensor $B_{ij}$ can be written as

$$T_{kl} B_{kl} = T_{kl}\left(S_{kl} + A_{kl}\right) = T_{kl} S_{kl} + T_k A_{kl} = T_{ij} S_{ij} + T_{ij} A_{ij}$$

## 2.12 Gauss' Theorem

For a given scalar, vector or tensor field $Q(\mathbf{x})$, Gauss' theorem states that

$$\iiint_V \frac{\partial Q}{\partial x_i} dV = \iint_A n_i Q dA$$

The most common form is when $Q$ is a vector field, leading to

$$\iiint_V \sum_{i=1}^{3} \frac{\partial Q_i}{\partial x_i} dV \equiv \iiint_V \frac{\partial Q_i}{\partial x_i} dV = \iint_A \sum_{i=1}^{3} n_i Q_i dA \equiv \iint_A n_i Q_i dA,$$

$$\text{or} \iiint_V \nabla \cdot Q dV = \iint_A \mathbf{n} \cdot Q dA,$$

# Chapter 3. Kinematics

## 3.1 Introduction and Coordinate Systems

**Kinematics** is the description of motion without reference to the forces or stresses that produce the motion.

**Time-dependence of flow**: When a flow changes with time, it is termed *unsteady*. When it does not it is called *steady*.

### 3.1.1 Dimensionality of Flow

A truly **one-dimensional flow** is one in which the flow's characteristics can be entirely described with one independent spatial variable.

A **two-dimensional**, or **plane**, flow is one in which the variation of flow characteristics can be described by two spatial coordinates.

A **three-dimensional flow** is one that can only be properly described with three independent spatial coordinates and is the most general case.

## 3.2 Particle and Field Descriptions of Fluid Motion

There are two ways to describe fluid motion:

- **Lagrangian description:** fluid particles are followed as they move through the flow field
- **Eulerian description:** flow field characteristics (e.g. fluid velocity) are monitored at fixed locations in space.

Understanding both is necessary, however, the Eulerian description is the favoured description in fluid dynamics (in contrast to solid mechanics, where the Lagrangian description is preferred).

### 3.2.1 Lagrangian Description

The Lagrangian description is based on the motion of fluid particles. The particle path $\mathbf{r}\left(t;\mathbf{r}_0,t_0\right)$ specifies the location of the fluid particle that started at location $\mathbf{r}_0$ at time $t_0$. Thus,

$$\mathbf{u} = d\mathbf{r}\left(t;\mathbf{r}_o,t_o\right)/dt \quad \text{and} \quad \mathbf{a} = d^2\mathbf{r}\left(t;\mathbf{r}_o,t_o\right)/dt^2 \tag{3.1}$$

Additionally, any flow property $F$ might depend on the path followed by the particle, where $F$ can be any scalar, vector or tensor property of the flow field.

$$F = F\left[\mathbf{r}\left(t;\mathbf{r}_0,t_0\right),t\right] \tag{3.2}$$

### 3.2.2 Eulerian Description

The Eulerian description focuses on flow field properties in fixed regions of interest and involves four independent variables: the three spatial coordinates represented by the position vector $\mathbf{x}$, and time $t$:

$$F = F(\mathbf{x},t) \tag{3.3}$$

To link the two descriptions we require equality of $F$ when $\mathbf{r}$ and $\mathbf{x}$ define same point in space:

$$F\left[\mathbf{r}\left(t;\mathbf{r}_o,t_o\right),t\right] = F(\mathbf{x},t) \text{ when } \mathbf{x} = \mathbf{r}\left(t;\mathbf{r}_o,t_o\right) \tag{3.4}$$

Here the second equation specifies the trajectory followed by a fluid particle.
Applying a total time derivative to the first equation produces:

$$\frac{d}{dt}F\left[\mathbf{r}\left(t;\mathbf{r}_o,t_o\right)t\right] = \frac{\partial F}{\partial r_1}\frac{dr_1}{dt} + \frac{\partial F}{\partial r_2}\frac{dr_2}{dt} + \frac{\partial F}{\partial r_3}\frac{dr_3}{dt} + \frac{\partial F}{\partial t} = \frac{d}{dt}F(\mathbf{x},t) \text{ when } \mathbf{x} = \mathbf{r}\left(t;\mathbf{r}_o,t_o\right) \tag{3.5}$$

The time derivatives of $r_i$ are the components $u_i$ of the fluid particle's velocity $\mathbf{u}$. When $\mathbf{x} = \mathbf{r}$, it holds that $\partial F/\partial r_i = \partial F/\partial x_i$, so the right side of can be rewritten entirely in the Eulerian description:

$$\frac{d}{dt}F\left[\mathbf{r}\left(t;\mathbf{r}_o,t_o\right),t\right] = \frac{\partial F}{\partial x_1}u_1 + \frac{\partial F}{\partial x_2}u_2 + \frac{\partial F}{\partial x_3}u_3 + \frac{\partial F}{\partial t} = (\nabla F)\cdot\mathbf{u} + \frac{\partial F}{\partial t} \equiv \frac{D}{Dt}F(\mathbf{x},t) \tag{3.6}$$

where the final equality defines $D/Dt$ as the total time derivative in the Eulerian description of fluid motion.

$$\frac{DF}{Dt} \equiv \frac{\partial F}{\partial t} + \mathbf{u}\cdot\nabla F \tag{3.7}$$

The **material derivative** $D/Dt$ is composed of unsteady and advective parts.

- $\partial F/\partial t$ is the unsteady part, it is the local temporal rate of change of $F$ at the location $\mathbf{x}$. It is zero when F is independent of time.
- $\mathbf{u}\cdot\nabla F$ is the advective (or convective) part, it is the rate of change of $F$ that occurs as fluid particles move from one location to another. It is zero where $F$ is spatially uniform, the fluid is not moving, or $\mathbf{u}$ and $\nabla F$ are perpendicular.

The total acceleration is given by:

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} \tag{3.8}$$

With $\partial\mathbf{u}/\partial t$ the local acceleration and $(\mathbf{u}\cdot\nabla)\mathbf{u}$ the advective acceleration.

## 3.3 Flow Lines, Fluid Acceleration, and Galilean Transformation

Streamlines, path lines and streak lines are curves that are used to describe fluid motion. When the flow is steady these curves are all the same.

### 3.3.1 Streamlines

A **streamline** is a curve that is instantaneously tangent to the fluid velocity throughout the domain. The tangency requirement on the arc length $d\mathbf{s} = (dx, dy, dz)$ and velocity $\mathbf{u} = (u, v, w)$ leads to

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \tag{3.9}$$

which needs to be integrated upstream and downstream from a reference point.

When the reference points for calculating the streamlines from the velocity profile $\mathbf{u}$ lie on a closed curve $C$, the resulting stream surface is called a **stream tube**. No fluid crosses a stream tube's surface since all velocity vectors are tangent to the surface.

### 3.3.2 Path Lines

A **path line** is the trajectory of a fluid particle of fixed identity:

$$\mathbf{x} = \mathbf{r}\left(t;\mathbf{r}_0,t_0\right) \tag{3.10}$$

The equation for the path line of a particle that was at location $\mathbf{r}_o$ at time $t_o$ can be obtained from $\mathbf{u}$ by integrating

$$d\mathbf{r}/dt = \mathbf{u}(\mathbf{r},t) \tag{3.11}$$

subject to $\mathbf{r}\left(t_o\right) = \mathbf{r}_0$.

Experimentally, path lines can be obtained by adding small particles to the fluid and tracking their location in time.

### 3.3.3 Streak Lines

A **streak line** is a curve obtained by connecting all the fluid particles that have passed through a fixed point in space. Streak lines may be visualized in experiments by injecting a passive marker, like dye or smoke, from a small port.

The streak line through the point $\mathbf{x}_0$ at time $t$ is found by integrating (3.11) for all relevant reference times, $t_0$, subject to the requirement $\mathbf{r}(t_0) = \mathbf{x}_0$. When completed, this integration provides a path line, $\mathbf{x} = \mathbf{r}(t; \mathbf{x}_0, t_0)$, for each value of to.

## 3.4 Strain and Rotation Rates

The basic constitutive law for fluids relates fluid element deformation rates to the stresses (surface forces per unit area) applied to a fluid element.

A three-dimensional first-order Taylor expansion of $\mathbf{u}$ about $\mathbf{x}$ leads to the velocity-gradient tensor:

$$du_i = (\partial u_i / \partial x_j)\, dx_j \tag{3.12}$$

This can be decomposed into:

$$\frac{\partial u_i}{\partial x_j} = S_{ij} + \frac{1}{2} R_{ij} \tag{3.13}$$

with the **strain rate tensor**:

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{3.14}$$

and the **rotation tensor**

$$R_{ij} = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \tag{3.15}$$

### 3.4.1 The Strain Rate Tensor

The strain rate tensor is given by:

$$[S_{ij}] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} \end{bmatrix} \tag{3.16}$$

The diagonal terms of the strain rate tensor, $S_{11}$ and $S_{22}$ represent elongation and contraction per unit length along the coordinate directions ("linear strain rates").

The off-diagonal terms of the strain rate tensor $S_{12} = S_{21}$ represent shear deformations that change the relative orientations of line segments, that were initially parallel to the coordinate directions.
Thus, the off-diagonal terms of $S_{ij}$ represent the average rate at which material line segments initially parallel to the $i$- and $j$-directions rotate *toward* each other.

### 3.4.2 The Rotation Tensor

The rotation tensor, $R_{ij}$ is antisymmetric so its diagonal elements are zero and its off-diagonal elements are equal and opposite. In 2D it is given by:

$$[R_{ij}] = \begin{bmatrix} 0 & \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} & 0 \end{bmatrix} \tag{3.17}$$

In 3D its three independent elements can be put in correspondence with a vector. This vector is the vorticity, $\omega = \nabla \times \mathbf{u}$, and the correspondence is:

$$R_{ij} = -\varepsilon_{ijk}\omega_k = \begin{array}{c} \text{2D} \\ \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \end{array}$$

where

$$\omega_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \ \ \omega_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \ \ \text{and } \omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \tag{3.18}$$

Fluid motion is called **irrotational** if

$$\omega = 0, \quad \text{or, equivalently,} \quad R_{ij} = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} = 0 \tag{3.19}$$

## 3.6 Reynolds Transport Theorem

Consider a moving volume $V^*(t)$ (called 'control volume'), having a closed surface $A^*(t)$ with outward normal $\mathbf{n}$ and let $\mathbf{b}$ denote the local velocity of $A^*$.

The time derivative of the integral of a continuous function $F(x,t)$ over this volume $V^*(t)$ is given by **Reynolds transport theorem**:

$$\frac{d}{dt}\int_{V^*(t)} F(\mathbf{x},t)dV = \int_{V^*(t)} \frac{\partial F(\mathbf{x},t)}{\partial t}dV + \int_{A^*(t)} F(\mathbf{x},t)\mathbf{b}\cdot\mathbf{n}dA. \tag{3.20}$$

This theorem is the basis to develop the integral and differential versions of the conservation laws of fluid motion.

# Chapter 4. Conservation Laws

## 4.2 Conservation of Mass

### 4.2.1 Integral Form

A statement of conservation of mass for a material volume in a flowing fluid is:

$$\frac{d}{dt}\int_{V(t)} \rho(\mathbf{x},t)dV = 0 \tag{4.1}$$

where $\rho$ is the fluid density and $V(t)$ is a **material volume**, which moves and deforms with the fluid flow so that it always contains the same mass.

From this follows the integral mass balance for a *material volume* $V(t)$:

$$\frac{d}{dt}\int_{V(t)} \rho(\mathbf{x},t)dV = \int_{V(t)} \frac{\partial \rho(\mathbf{x},t)}{\partial t}dV + \int_{A(t)} \rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)\cdot\mathbf{n}dA = 0 \tag{4.2}$$

And the integral mass balance for an *arbitrary control volume* $V^*(t)$ having velocity $\mathbf{b}$:

$$\frac{d}{dt}\int_{V^*(t)} \rho(\mathbf{x},t)dV + \int_{A^*(t)} \rho(\mathbf{x},t)(\mathbf{u}(\mathbf{x},t)-\mathbf{b})\cdot\mathbf{n}dA = 0 \tag{4.3}$$

Substituting $\mathbf{b} = \mathbf{u}$ in Eq. (4.3) retrieves Eq. (4.2). A fixed control volume corresponds to $\mathbf{b} = 0$.

### 4.2.2 Differential Form

The **continuity equation** expresses the principle of conservation of mass in differential form, this is given by:

$$\frac{\partial \rho(\mathbf{x},t)}{\partial t} + \nabla\cdot(\rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)) = 0 \quad \text{or,} \quad \text{in index notation: } \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \tag{4.4}$$

Two special cases of the continuity equation are:

1. Steady, compressible flow
$$\nabla\cdot(\rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)) = 0 \tag{4.5}$$
2. Incompressible flow (both steady and unsteady)
$$\nabla\cdot\mathbf{u} = 0 \tag{4.6}$$

Note that Eq. (4.6) corresponds to setting the volumetric strain rate to be equal to zero.
A fluid is usually called incompressible when its density $\rho$ does not change with pressure.

## 4.4 Conservation of Momentum

### 4.4.1 Integral Form

When applied to a material volume $V(t)$ with surface area $A(t)$, Newton's second law gives the momentum conservation which can be stated directly as:

$$\frac{d}{dt}\int_{V(t)} \rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)dV = \int_{V(t)} \rho(\mathbf{x},t)\mathbf{g}dV + \int_{A(t)} \mathbf{f}(\mathbf{n},\mathbf{x},t)dA, \tag{4.7}$$

where $\rho\mathbf{u}$ is the momentum per unit volume of the flowing fluid, $\mathbf{g}$ is the body force per unit mass acting on the fluid within $V(t)$, $\mathbf{f}$ is the surface force per unit area acting on $A(t)$.

From this follows the integral momentum balance for a *material volume* $V(t)$:

$$\begin{aligned}\int_{V(t)} \frac{\partial}{\partial t}(\rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t))dV + \int_{A(t)} \rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)(\mathbf{u}(\mathbf{x},t)\cdot\mathbf{n})dA \\ = \int_{V(t)} \rho(\mathbf{x},t)\mathbf{g}dV + \int_{A(t)} \mathbf{f}(\mathbf{n},\mathbf{x},t)dA\end{aligned} \tag{4.8}$$

And the integral momentum balance for an *arbitrary control volume* $V^*(t)$ having velocity $\mathbf{b}$:

$$\begin{aligned}\frac{d}{dt}\int_{V^*(t)} \rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)dV + \int_{A^*(t)} \rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)(\mathbf{u}(\mathbf{x},t)-\mathbf{b})\cdot\mathbf{n}dA \\ = \int_{V^*(t)} \rho(\mathbf{x},t)\mathbf{g}dV + \int_{A^*(t)} \mathbf{f}(\mathbf{n},\mathbf{x},t)dA\end{aligned} \tag{4.9}$$

The **body force**, $\rho\mathbf{g}dV$, acts on the fluid element $dV$ without physical contact. Body forces commonly arise from gravitational, magnetic, electrostatic, or electromagnetic force fields (or fictitious body forces due to non-inertial motion).
Body forces may be conservative or nonconservative. *Conservative body forces* are those that can be expressed as the gradient of a potential function:

$$\mathbf{g} = -\nabla\Phi \quad \text{or} \quad g_j = -\partial\Phi/\partial x_j \tag{4.10}$$

where $\Phi$ is called the *force potential*; it has units of energy per unit mass.

Surface forces, $\mathbf{f}$, act on fluid elements through direct contact with the surface of the element. They are proportional to the contact area and carry units of stress (force per unit area).

### 4.4.2 Bernoulli Equation

Using a stream-tube control volume of differential length $ds$ for a steady, inviscid, constant density flow where $U$ is the local flow speed, the **Bernoulli equation** can be found.

$$U^2/2 + gz + p/\rho = \text{ a constant along a streamline.} \tag{4.11}$$

Or the unintegrated versions:

$$U\frac{\partial U}{\partial s}ds = -gdz - \frac{1}{\rho}\frac{\partial p}{\partial s}ds, \quad \text{or} \quad \left[d\left(U^2/2\right) + gdz + (1/\rho)dp = 0\right]_{\text{along a streamline}} \tag{4.12}$$

The Bernoulli equation can also be written in units of length:

$$\frac{p}{\gamma} + \frac{V^2}{2g} + z = \text{ constant on a streamline } = H \tag{4.13}$$

With:

- $p/\gamma = p/\rho g$: Pressure head, the column height to reach $p$
- $V^2/2g$: Velocity head, the vertical drop distance from rest to reach $V$ (in absence of friction)
- $z$: Elevation head, the potential energy
- $H$: Total head

### 4.4.3 Differential Form

**Cauchy's equation of motion** expresses the principle of conservation of momentum in differential form, this is given by:

$$\rho\frac{Du_j}{Dt} = \rho g_j + \frac{\partial}{\partial x_i}(\tau_{ij}) \tag{4.14}$$

(4.14) relates fluid-particle acceleration to the net body ($\rho g_j$) and surface $\partial\tau_{ij}\partial x_i$ forces on the particle. It is true in any continuum, solid or fluid, no matter how the stress tensor $\tau_{ij}$ is related to the velocity field. However, (4.14) does not provide a complete description of fluid dynamics, even when combined with (4.4) because the number of dependent field variables is greater than the number of equations (13 unknowns vs 6 equations).

## 4.5 Constitutive Equation for a Newtonian Fluid

The stress tensor is given by:

$$\tau = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \tag{4.15}$$

The diagonal elements are the normal stresses, and the off-diagonal elements are the tangential or shear stresses. Considering the torque produced by these stresses, it can be shown that the stress tensor is symmetric,

$$\tau_{ij} = \tau_{ji} \tag{4.16}$$

then the number of independent stress components reduces from 9 to 6.

In a fluid *at rest*, the stress is **isotropic**:

$$\tau_{ij} = -p\delta_{ij} \tag{4.17}$$

where $p$ is the thermodynamic pressure related to $\rho$ and $T$ by an equation of state such as that for a perfect gas $p = \rho RT$.

A moving fluid develops fluid-dynamic contributions to the stress tensor, called the *deviatoric stresses* due to viscosity, so that

$$\tau_{ij} = -p\delta_{ij} + \sigma_{ij}. \tag{4.18}$$

Deviatoric stress only develops in fluid elements that change shape, so only the symmetric part of the velocity gradient tensor ($S_{ij}$) should enter.
The most general linear relation between $\sigma_{ij}$ and $S_{mn}$ can be written as:

$$\sigma_{ij} = K_{ijmn}S_{mn} \tag{4.19}$$

where $K_{ijmn}$ is a 4th-order tensor which has 81 components.
When the fluid is *isotropic* the stress-strain rate relationship is independent of the orientation of the coordinate system and the $K_{ijmn}$ becomes:

$$K_{ijmn} = \lambda\delta_{ij}\delta_{mn} + 2\mu\delta_{im}\delta_{jn} \tag{4.20}$$

Then the stress becomes:

$$\tau_{ij} = -p\delta_{ij} + 2\mu S_{ij} + \lambda S_{mm}\delta_{ij} \tag{4.21}$$

By setting $i = j$ this can be solved for the pressure:

$$p - \bar{p} = \left(\frac{2}{3}\mu + \lambda\right)\nabla\cdot\mathbf{u} \qquad \bar{p} \equiv -\frac{1}{3}\tau_{ii} \tag{4.22}$$

with the *coefficient of bulk viscosity* defined as: $\mu_v = \lambda + 2\mu/3$.

Additionally, a general equation for an isotropic compressible, Newtonian fluid can be found:

$$\tau_{ij} = -p\delta_{ij} + 2\mu\left(S_{ij} - \frac{1}{3}S_{mm}\delta_{ij}\right) + \mu_v S_{mm}\delta_{ij} \tag{4.23}$$

And for an incompressible fluid, the stress is:

$$\tau_{ij} = -p\delta_{ij} + 2\mu S_{ij} \tag{4.24}$$

This linear relation between $\tau$ and $\mathbf{S}$ is consistent with Newton's definition of the viscosity coefficient $\mu$ in a simple parallel flow $u(y)$.
For non-Newtonian fluids, shear stress and shear strain rate are not linearly related. Then, viscosity can be defined as the instantaneous slope of the shear stress/shear strain rate curve.

## 4.6 Navier-Stokes Momentum Equation

Substituting the general isotropic Newtonian constitutive equation (4.23) into Cauchy's equation (4.14) yields the **Navier-Stokes momentum equation**:

$$\rho \frac{Du_j}{Dt} = -\frac{\partial p}{\partial x_j} + \rho g_j + \frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) + \left(\mu_v - \frac{2}{3}\mu\right)\frac{\partial u_m}{\partial x_m}\delta_{ij}\right] \tag{4.25}$$

with the viscosities $\mu$ and $\mu_v$ only depending on the thermodynamic state.

Together with the continuity equation, there are now have 4 equations for 5 unknowns ($\rho$, $u_i$ and $p$). For constant $\rho$ (incompressible fluids) or when $\rho = \rho(p)$ is known, we have a *complete description of fluid dynamics.*

For incompressible fluids and uniform $\mu$ the fluid dynamics equations reduce to the *incompressible Navier-Stokes equation*

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{u} \tag{4.26}$$

When viscous effects are negligible, the Navier-Stokes equation simplifies into the **Euler equation**:

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{g} \tag{4.27}$$

## 4.8 Conservation of Energy

When the fluid is compressible and when the relationship between $\rho$ and $p$ also includes $T$, the internal energy $e$ needs to be considered. Then, also conservation of energy is needed to obtain a full description of fluid motion.

### 4.8.1 Integral Form

When applied to a *material volume* $V(t)$ with surface area $A(t)$, the conservation of internal energy for a single-component fluid can be stated:

$$\frac{d}{dt}\int_{V(t)} \rho\left(e + \frac{1}{2}|\mathbf{u}|^2\right) dV = \int_{V(t)} \rho \mathbf{g}\cdot\mathbf{u} dV + \int_{A(t)} \mathbf{f}\cdot\mathbf{u} dA - \int_{A(t)} \mathbf{q}\cdot\mathbf{n} dA \tag{4.28}$$

where the terms on the right are:

- work done on the fluid in $V(t)$ by body forces ($\mathbf{g}$),
- work done on the fluid in $V(t)$ by surface forces ($\mathbf{f}$),
- heat transferred out of $V(t)$ (with $\mathbf{q}$ the heat flux).

This can be generalized to an *arbitrarily moving control volume* $V^*(t)$ with surface $A^*(t)$:

$$\begin{aligned} &\frac{d}{dt}\int_{V^*(t)} \rho\left(e + \frac{1}{2}|\mathbf{u}|^2\right) dV + \int_{A^*(t)} \left(\rho e + \frac{\rho}{2}|\mathbf{u}|^2\right)(\mathbf{u} - \mathbf{b})\cdot\mathbf{n} dA \\ &= \int_{V^*(t)} \rho \mathbf{g}\cdot\mathbf{u} dV + \int_{A^*(t)} \mathbf{f}\cdot\mathbf{u} dA - \int_{A^*(t)} \mathbf{q}\cdot\mathbf{n} dA, \end{aligned} \tag{4.29}$$

### 4.8.2 Differential Form

The differential statement of energy conservation is given by:

$$\frac{De}{Dt} = -p\frac{Dv}{Dt} + \frac{1}{\rho}\sigma_{ij}S_{ij} - \frac{1}{\rho}\frac{\partial q_i}{\partial x_i} \tag{4.30}$$

Here, $v = 1/\rho$.

### 4.8.3 A Full Description

Assuming that the fluid is Newtonian and follows Fourier's law of heat conduction ($\mathbf{q} = -k\nabla T$) yields the differential energy balance as

$$\rho \frac{De}{Dt} = -p\frac{\partial u_m}{\partial x_m} + 2\mu \left( S_{ij} - \frac{1}{3}\frac{\partial u_m}{\partial x_m}\delta_{ij} \right)^2 + \mu_v \left( \frac{\partial u_m}{\partial x_m} \right)^2 + \frac{\partial}{\partial x_i}\left( k\frac{\partial T}{\partial x_i} \right) \tag{4.31}$$

which, together with the continuity equation, the Navier-Stokes momentum equation and two thermodynamic equations of state

$$p = p(v, T), \quad e = e(p, T) \tag{4.32}$$

provide a full description of fluid motion for a Newtonian fluid which follows Fourier's law of heat conduction.

## 4.9 Special Forms of the Equations

### 4.9.1 Applications of Bernoulli equations

**Pitot tube.** Consider first a simple device to measure the local velocity in a fluid stream by inserting a narrow bent tube, called a *pitot tube*. Consider two points (1 and 2) at the same level, point 1 being away from the tube and point 2 being immediately in front of the open end where the fluid velocity $\mathbf{u}_2$ is zero. If the flow is steady and irrotational with constant density along the streamline that connects 1 and 2, then (4.11) gives:

$$\frac{p_1}{\rho} + \frac{1}{2}\left|\mathbf{u}_1\right|^2 = \frac{p_2}{\rho} + \frac{1}{2}\left|\mathbf{u}_2\right|^2 = \frac{p_2}{\rho} \tag{4.33}$$

from which the magnitude of $\mathbf{u}_1$ is found to be:

$$|\mathbf{u}_1| = \sqrt{2\left(p_2 - p_1\right)/\rho}. \tag{4.34}$$

Pressures at the two points are found from the hydrostatic balance:

$$p_1 = \rho g h_1 \quad \text{and } p_2 = \rho g h_2, \tag{4.35}$$

so that the magnitude of $\mathbf{u}_1$ can be found from:

$$|\mathbf{u}_1| = \sqrt{2g\left(h_2 - h_1\right)}. \tag{4.36}$$

The pressure $p_2$ measured by a pitot tube is called stagnation pressure or total pressure, which is larger than the local static pressure.

**Free jet.** As another application of Bernoulli's equation, consider the flow through an orifice or opening in a tank. The flow is slightly unsteady due to the lowering of the water level in the tank, but this effect is small if the tank area is large compared to the orifice area. Viscous effects are negligible everywhere away from the walls of the tank. All streamlines can be traced back to the free surface in the tank, where they have the same value of the Bernoulli constant $B = |\mathbf{u}|^2/2 + p/\rho + gz$. It follows that the flow is irrotational, and $B$ is constant throughout the flow.

Application of the Bernoulli equation (4.11) for steady constant-density flow between a point on the free surface in the tank and a point in the jet:

$$\frac{p_{atm}}{\rho} + gh = \frac{p_{atm}}{\rho} + \frac{u^2}{2},$$

from which the average jet velocity magnitude $u$ is found as:

$$u = \sqrt{2gh}$$

which simply states that the loss of potential energy equals the gain of kinetic energy.

The mass flow rate in the jet is approximately: $\dot{m} = \rho A_c u = \rho A_c\sqrt{2gh}$, where $A_c$ is the area of the jet. For orifices having a sharp edge, $A_c$ has been found to be $\approx 62\%$ of the orifice area because the jet contracts downstream of the orifice opening.

## 4.10 Boundary Conditions

Applying the integral form of mass conservation to a small cylindrical control volume with thickness $l$ specifies $\rho_1 \mathbf{u}_1 \cdot \mathbf{n} = \rho_2 \mathbf{u}_2 \cdot \mathbf{n}$ between fluids 1 and 2.
When medium 2 is a solid we get

$$\mathbf{u}_1 \cdot \mathbf{n} = 0 \tag{4.37}$$

on the boundary. Applying the integral form of momentum conservation to the control volume specifies that the tractions

$$n_i \tau_{ij}, \tag{4.38}$$

are continuous. Finally, energy conservation specifies that the heat flux

$$n_i q_i \tag{4.39}$$

must be continuous across the interface. These equations have to be supplemented with zero temperature jump $T_1 = T_2$ and with the no-slip condition when medium 2 is a solid:

$$\mathbf{u}_1 \cdot \mathbf{t} = 0 \tag{4.40}$$

## 4.11 Dimensionless Forms of the Equations and Dynamic Similarity

The dimensionless parameters for any particular problem can be determined in two ways.

1. From the system parameters (e.g. using Buckingham Π-theorem), if e.g. the differential equations are not known,
2. Directly from the governing differential equations.

Defining dimensionless variables allows for rendering the governing differential equations dimensionless:

$$x_i^* = x_i/l, \quad t^* = \Omega t, \quad u_j^* = u_j/U, \quad p^* = (p - p_\infty)/\rho U^2, \quad \text{and} \quad g_j^* = g_j/g, \tag{4.41}$$

Depending on the nature of the flow, $p - p_\infty$ could be made dimensionless with a generic viscous stress $\mu U/l$, a hydrostatic pressure $\rho g l$, or a dynamic pressure $\rho U^2$.

Using these dimensionless variables, the incompressible Navier-Stokes equation (4.26) becomes:

$$\left[\frac{\Omega l}{U}\right] \frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*)\,\mathbf{u}^* = -\nabla^* p^* + \left[\frac{gl}{U^2}\right] \mathbf{g}^* + \left[\frac{\mu}{\rho U l}\right] \nabla^{*2} \mathbf{u}^* \tag{4.42}$$

where $\nabla^* = l\nabla$.

The parameter groupings shown in [,]-brackets have the following names and interpretations:

$$\mathrm{St} = \text{ Strouhal number } \equiv \frac{\text{unsteady acceleration}}{\text{advective acceleration}} \propto \frac{\partial u/\partial t}{u(\partial u/\partial x)} \propto \frac{\Omega U}{U^2/l} = \frac{\Omega l}{U}, \tag{4.43}$$

$$\mathrm{Re} = \text{ Reynolds number } \equiv \frac{\text{inertia force}}{\text{viscous force}} \propto \frac{\rho u(\partial u/\partial x)}{\mu\,(\partial^2 u/\partial x^2)} \propto \frac{\rho U^2/l}{\mu U/l^2} = \frac{\rho U l}{\mu}, \quad \text{and} \tag{4.44}$$

$$\mathrm{Fr} = \text{ Froude number } \equiv \left[\frac{\text{inertia force}}{\text{gravity force}}\right]^{1/2} \propto \left[\frac{\rho u(\partial u/\partial x)}{\rho g}\right]^{1/2} \propto \left[\frac{\rho U^2/l}{\rho g}\right]^{1/2} = \frac{U}{\sqrt{gl}}. \tag{4.45}$$

It follows that in dynamically similar flows, dimensionless flow variables are identical at corresponding points and times (that is, for identical values of $x/1$, and $\Omega t$):

$$\frac{p(\mathbf{x}, t) - p_\infty}{\frac{1}{2}\rho U^2} \equiv C_p = \Psi\left(\mathrm{St}, \mathrm{Fr}, \mathrm{Re}; \frac{\mathbf{x}}{l'}, \Omega t\right), \tag{4.46}$$

with $C_p$ defined as the *pressure coefficient* (or **Euler number**) and $\Psi$ is the dimensionless solution.

All dimensionless quantities are identical in dynamically similar flows. For flow around an immersed body, like a sphere, we can define the (dimensionless) drag and lift coefficients:

$$C_D \equiv \frac{F_D}{\frac{1}{2}\rho U^2 A} \quad \text{and} \quad C_L \equiv \frac{F_L}{\frac{1}{2}\rho U^2 A} \tag{4.47}$$

# Chapter 7. Ideal Flow

## 7.1 Relevance of Irrotational Constant-Density Flow Theory

For *incompressible* fluids (and constant $\mu$), the fluid dynamics equations that are provided by the incompressible Navier-Stokes and the continuity equations reduce to

$$\nabla \cdot \mathbf{u} = 0 \text{ and } \rho(D\mathbf{u}/Dt) = -\nabla p \tag{7.1}$$

These are the equations of **ideal flow**.

### 7.1.1 Bernoulli Equation for Ideal Flow

The Bernoulli equation for ideal flow is derived from (7.1) and is given by:

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \nabla \left( \frac{1}{2}\rho |\mathbf{u}|^2 + p \right) = 0 \tag{7.2}$$

Two simplifications for this equation can be made. First, when the flow is steady, the first term drops and we can write:

$$\frac{1}{2}\rho |\mathbf{u}|^2 + p = \text{ constant.} \tag{7.3}$$

The big difference between the earlier Bernoulli equation is that now we require the flow to be irrotational, which causes the constant to be the same throughout the flow (*so not only along a streamline*).

Second, when the flow is unsteady, we write the velocity as the gradient of a scalar potential $\phi$ (called the velocity potential):

$$\mathbf{u} \equiv \nabla \phi \tag{7.4}$$

Then the Bernoulli equations can be written as:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 + \frac{p}{\rho} = \text{ constant.} \tag{7.5}$$

This is the unsteady Bernoulli equation for incompressible, irrotational flow, where, again, the constant is equal *throughout the entire flow.*
An important consequence is that if the flow is irrotational in a certain region, it remains irrotational throughout the flow as long as viscous effects are negligible.

### 7.1.2 Application of Ideal Flow Theory

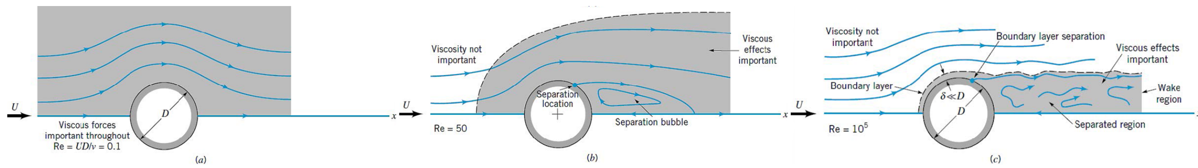
Ideal flow theory has abundant applications in the *exterior* aero- and hydrodynamics of moderate- to large-scale objects at non-trivial subsonic speeds. Here, *moderate size* ($L$) and *non-trivial speed* ($U$) are determined jointly by the requirement that the Reynolds number, $\text{Re} = \rho U l/\mu$, be large enough (typically $\text{Re} \sim 10^3$ or greater) so that the combined influence of fluid viscosity and fluid element rotation is confined to thin layers on solid surfaces, commonly known as **boundary layers** or fluid particle accelerations caused by fluid inertia $\sim U^2/L$ are much larger than those caused by viscosity $\sim \mu U/\rho L^2$.

Because (7.1) involves only first-order spatial derivatives, ideal flows only satisfy the no-through-flow boundary condition on solid surfaces. The no-slip boundary condition is not applied in ideal flows, so non-zero tangential velocity at a solid surface may exist. As a consequence of the fact that ideal flow cannot predict viscous effects such as skin friction, it cannot be applied to the interior flow in pipes and ducts (thus only to the exterior of objects).

At sufficiently high Re, there are two primary differences between ideal and real flows over the same object.

1. Viscous boundary layers containing rotational fluid develop on solid surfaces in the real flow, and the thickness of such boundary layers, within which viscous diffusion of vorticity is important, approaches zero as $\text{Re} \to \infty$.

2. The possible formation in the *real* flow of separated flow or wake regions that occur when boundary layers leave the surface on which they have developed to create a wider zone of rotational flow.




## 7.2 Two-Dimensional Stream Function and Velocity Potential

The two-dimensional incompressible continuity equation:

$$\partial u/\partial x + \partial v/\partial y = 0 \tag{7.6}$$

is identically satisfied when $u, v$-velocity components are determined from a single scalar function $\psi$:

$$u \equiv \partial\psi/\partial y, \quad \text{and} \quad v \equiv -\partial\psi/\partial x \tag{7.7}$$

The function $\psi(x, y)$ is the **stream function** in two dimensions. Along a curve of $\psi =$ constant, $d\psi = 0$, and this implies:

$$0 = d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = -vdx + udy, \text{ or } \left(\frac{dy}{dx}\right)_{\psi=\text{ const}} = \frac{v}{u'} \tag{7.8}$$

which is the definition of a streamline in two dimensions. From the vorticity $\omega_z$ of a flow in 2D we know that:

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \omega_z = \frac{\partial}{\partial x}\left(-\frac{\partial\psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial y}\right) = -\nabla^2\psi = 0 \tag{7.9}$$

*Alternatively*, the condition of irrotationality in two dimensions is:

$$\partial v/\partial x - \partial u/\partial y = 0 \tag{7.10}$$

and it is identically satisfied when $u, v$-velocity components are determined from a single scalar function $\phi$:

$$u \equiv \partial\phi/\partial x, \quad \text{and} \quad v \equiv \partial\phi/\partial y \tag{7.11}$$

The function f(x,y) is known as the **velocity potential** in two dimensions because $\nabla\phi = \mathbf{u}$. In fact, a velocity potential must exist in all irrotational flows, so such flows are frequently called potential flows. Curves of $\phi =$ constant are defined by:

$$0 = d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy = udx + vdy \quad \text{or} \quad \left(\frac{dy}{dx}\right)_{\phi=\text{ const}} = -\frac{u}{v} \tag{7.12}$$

which are **equipotential lines** that are perpendicular to the streamlines. The condition of incompressibility now becomes:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right)\phi = \nabla^2\phi = 0 \tag{7.13}$$

### 7.2.1 Boundary Conditions

The boundary conditions normally encountered in irrotational flows are as follows.

1. No flow through a stationary solid surface:

$$(\mathbf{n}\cdot\mathbf{u})_{\text{on the surface}} = 0 \tag{7.14}$$

which implies

$$\mathbf{n}\cdot\nabla\phi = \partial\phi/\partial n = 0 \text{ or } \partial\psi/\partial s = 0 \text{ on the surface,} \tag{7.15}$$

where $s$ is the arc length along the surface, and $n$ is the surface-normal coordinate. However, $\partial\psi/\partial s$ is also zero along a streamline. *Thus, a stationary solid boundary in an ideal flow must also be a streamline.*

2. Recovery of conditions at infinity. For the typical case of a body immersed in a uniform fluid flowing in the $x$ direction with speed $U$, the condition far from the body is

$$\partial\phi/\partial x = U, \text{ or } \partial\psi/\partial y = U \tag{7.16}$$

### 7.2.2 Laplace and Bernoulli

A solution to the Laplace equations has been obtained:

$$\nabla^2\psi = 0 \quad \text{or} \quad \nabla^2\phi = 0 \tag{7.17}$$

the velocity components are determined by

$$u \equiv \partial\psi/\partial y, \text{ and } v \equiv -\partial\psi/\partial x, \quad \text{or} \quad u \equiv \partial\phi/\partial x, \text{ and } v \equiv \partial\phi/\partial y \tag{7.18}$$

Then, the conservation of mass and momentum equations

$$\nabla\cdot\mathbf{u} = 0 \quad \text{and} \quad \rho\frac{\partial\mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}\rho|\mathbf{u}|^2 + p\right) = 0 \tag{7.19}$$

are satisfied. For steady flow, the pressures follow from the Bernoulli equation:

$$p + \frac{1}{2}\rho|\mathbf{u}|^2 = p + \frac{1}{2}\rho\left(u^2+v^2\right) = p + \frac{1}{2}\rho|\nabla\phi|^2 = p + \frac{1}{2}\rho|\nabla\psi|^2 = \text{ const.} \tag{7.20}$$

For unsteady flow, the term $\partial\phi/\partial t$ must be added.

## 7.3 Construction of Elementary Flows in Two Dimensions

### 7.3.1 Uniform flow

In an unbounded domain, the most elementary solutions of $\nabla^2\psi = 0$ and $\nabla^2\phi = 0$ are first-order polynomials for the stream function $\psi$ and velocity potential $\phi$:

$$\psi = -Vx + Uy, \quad \text{and} \quad \phi = Ux + Vy \tag{7.21}$$

These correspond to uniform fluid velocity with horizontal component $U$ and vertical component $V$. Uniform flow at an angle $\alpha$ is given by:

$$\begin{aligned}\psi &= U(y\cos\alpha - x\sin\alpha)\\ \phi &= U(x\cos\alpha + y\sin\alpha)\end{aligned} \tag{7.22}$$

### 7.3.2 Corner Flow

Ideal flow in a corner is given by second-order polynomials. For the stream function, this gives:

$$\psi = 2Axy \quad \longrightarrow \quad u = 2Ax, \text{ and } v = -2Ay \tag{7.23}$$

for $A > 0$, the flow is toward the origin along the $y$-axis, and away from it along the $x$-axis. The streamlines ($\psi$ = constant) are hyperbolae given by:

$$xy = \psi/2A \tag{7.24}$$

Considering the first quadrant only, this is the ideal flow in a 90° corner.

Considering the velocity potential yields:

$$\phi = 2Axy \quad \longrightarrow \quad u = 2Ay, \text{ and } v = 2Ax \tag{7.25}$$

The equipotential lines ($\phi$ = constant) are hyperbolae given by:

$$xy = \phi/2A \tag{7.26}$$

The flow is away from the origin along the line $y = x$ and toward it along the line $y = -x$. Thus, $\phi = 2Axy$ produces a flow that is equivalent to that of $\psi = 2Axy$ after a 45° rotation. Higher-order polynomial solutions lead to flows in smaller-angle corners, while fractional powers lead to flows in larger-angle corners.

In polar form, the results can be generalized to describe flow in the vicinity of a corner of angle $\alpha$ with the equation

$$\psi = Ar^{\pi/\alpha} \sin \frac{\pi\theta}{\alpha} \tag{7.27}$$

### 7.3.3 Ideal Irrotational Flow

Other than uniform flow, the most elementary solution of $\nabla^2\psi = 0$ in an unbounded domain is:

$$\psi = -\frac{\Gamma}{2\pi} \ln \sqrt{(x - x')^2 + (y - y')^2} \tag{7.28}$$

Which represents the flow induced by an irrotational vortex of strength $\Gamma$ located at $\mathbf{x}' = (x', y')$.

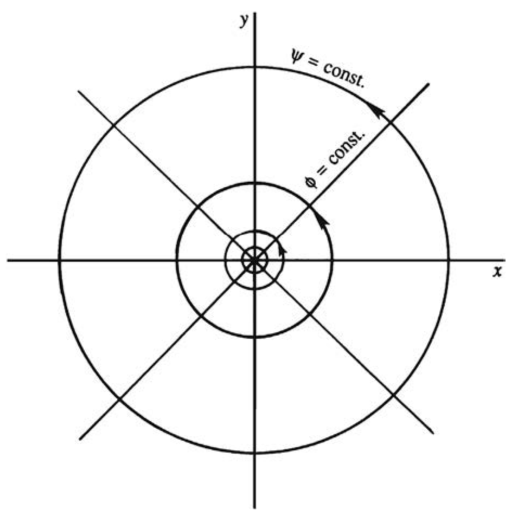
For $x' = y' = 0$ the velocities can be written as

$$\begin{aligned} u &= \frac{\partial}{\partial y}\left(-\frac{\Gamma}{2\pi} \ln \sqrt{x^2 + y^2}\right) = -\frac{\Gamma}{2\pi} \frac{y}{x^2 + y^2} = -\frac{\Gamma}{2\pi r} \sin\theta, \quad \text{and} \\ v &= -\frac{\partial}{\partial x}\left(-\frac{\Gamma}{2\pi} \ln \sqrt{x^2 + y^2}\right) = +\frac{\Gamma}{2\pi} \frac{x}{x^2 + y^2} = \frac{\Gamma}{2\pi r} \cos\theta \end{aligned} \tag{7.29}$$

where $r$ and $\theta$ are polar coordinates, related to $x$ and $y$ by $x = r\cos\theta, y = r \sin\theta$. With

$$u_r = 0 \text{ and } u_\theta = \Gamma/2\pi r, \tag{7.30}$$

which is the flow field of an ideal irrotational vortex.




The streamlines of an irrotational vortex are circles and the equipotential lines are radials from the origin.

### 7.3.4 Point Source

Other than uniform flow, the most elementary solution of $\nabla^2\phi = 0$ in an unbounded domain is:

$$\phi = \frac{q_s}{2\pi} \ln \sqrt{(x - x')^2 + (y - y')^2} \tag{7.31}$$

Which represents the flow induced by an ideal point source of strength $q_s$ located at $\mathbf{x}' = (x', y')$. Here, $q_s$ is the source's volume flow rate per unit length perpendicular to the plane of the flow.

For $x' = y' = 0$ the velocities can be written as

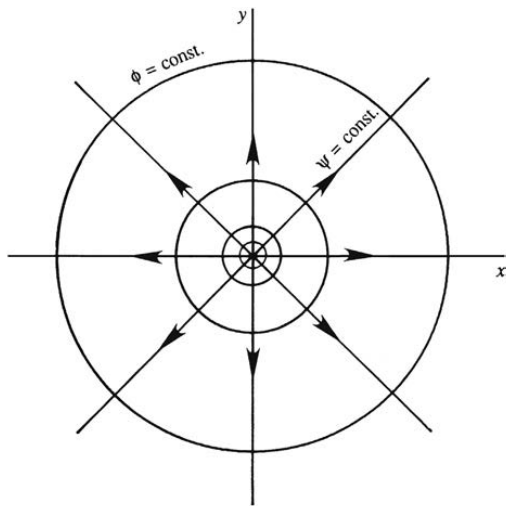
$$\begin{aligned} u &= \frac{\partial}{\partial x}\left(\frac{q_s}{2\pi}\ln\sqrt{x^2+y^2}\right) = \frac{q_s}{2\pi}\frac{x}{x^2+y^2} = \frac{q_s}{2\pi r}\cos\theta, \quad \text{and} \\ v &= \frac{\partial}{\partial y}\left(\frac{q_s}{2\pi}\ln\sqrt{x^2+y^2}\right) = \frac{q_s}{2\pi}\frac{y}{x^2+y^2} = \frac{q_s}{2\pi r}\sin\theta. \end{aligned} \tag{7.32}$$

These results may be written instead in polar coordinates as:

$$u_r = q_s/2\pi r \text{ and } u_\theta = 0, \tag{7.33}$$

from which it directly follows that the divergence is zero:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0 \tag{7.34}$$




The equipotential lines of an ideal point source are circles and the streamlines are radials from the origin. Thus, this potential represents the flow from an ideal incompressible point source for $q_s > 0$, or sink for $q_s < 0$, that is located at $r = 0$ in two dimensions.

### 7.3.5 Doublet (a Source and a Sink)

A source of strength $+q_s$ at $(-\varepsilon, 0)$ and sink of strength $-q_s$ at $(+\varepsilon, 0)$, can be considered together

$$\phi = \frac{q_s}{2\pi}\ln\sqrt{(x+\varepsilon)^2+y^2} - \frac{q_s}{2\pi}\ln\sqrt{(x-\varepsilon)^2+y^2} \tag{7.35}$$

to obtain the potential for a *doublet* in the limit that $\varepsilon \to 0$ and $q_s \to \infty$, so that the dipole strength vector:

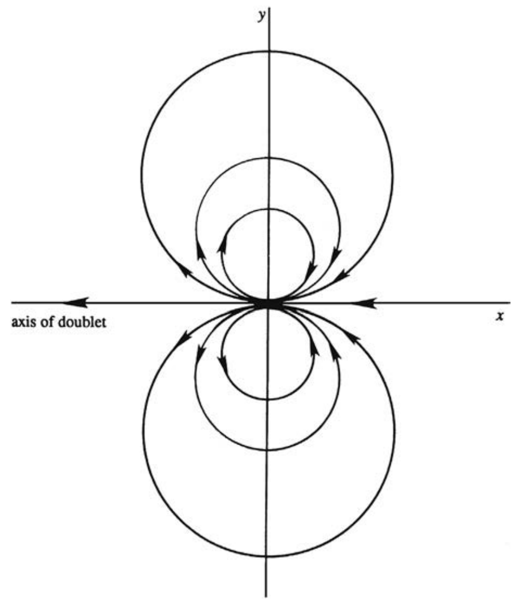
$$\mathbf{d} = \sum_{\text{sources}} \mathbf{x}_i q_{s,i} = -\varepsilon\mathbf{e}_x q_s + \varepsilon\mathbf{e}_x(-q_s) = -2q_s\varepsilon\mathbf{e}_x \tag{7.36}$$

remains constant. Here, the dipole strength points from the sink toward the source. In the limit $\varepsilon \to 0$ and $q_s \to \infty$ the velocity potential can be written in terms of the dipole vector d:

$$\lim_{\varepsilon\to 0}\lim_{q_s\to\infty}\phi \cong \frac{q_s\varepsilon}{\pi}\frac{x}{r^2} = -\frac{\mathbf{d}\cdot\mathbf{x}}{2\pi r^2} = \frac{|\mathbf{d}|}{2\pi}\frac{\cos\theta}{r} \tag{7.37}$$

The stream function of a doublet can similarly be derived, resulting in

$$\lim_{\varepsilon\to 0}\lim_{q_s\to\infty}\psi \cong -\frac{q_s\varepsilon}{\pi}\frac{y}{r^2} = -\frac{|\mathbf{d}|}{2\pi}\frac{\sin\theta}{r} \tag{7.38}$$




## 7.4 Superposition of Elementary Flows

The most common and useful superposition of these solutions involves combining a uniform stream parallel to the $x$-axis, $\psi = Uy$ or $\phi = Ux$, and one or more of the singular solutions.

### 7.4.1 Uniform Flow + Point Source

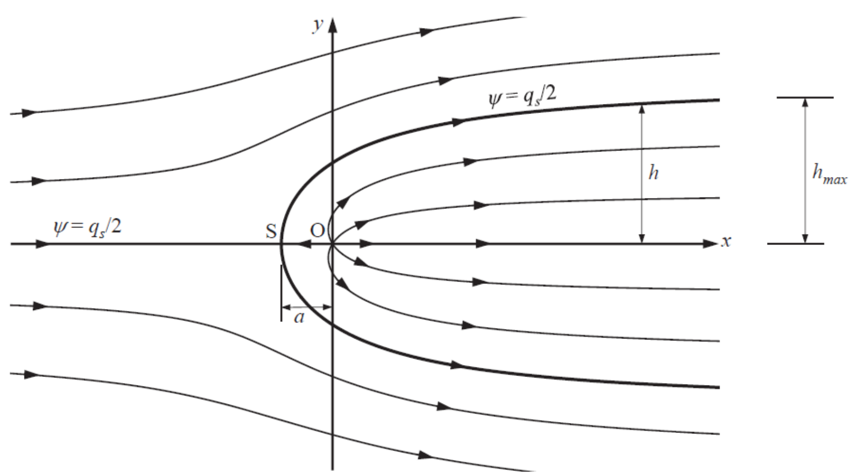
The simplest example is the combination of a source and a uniform stream, which can be written in Cartesian and polar coordinates as:

$$\begin{aligned} \phi &= Ux + \frac{q_s}{2\pi}\ln\sqrt{x^2+y^2} = Ur\cos\theta + \frac{q_s}{2\pi}\ln r, \quad \text{or} \\ \psi &= Uy + \frac{q_s}{2\pi}\tan^{-1}\left(\frac{y}{x}\right) = Ur\sin\theta + \frac{q_s}{2\pi}\theta. \end{aligned} \tag{7.39}$$

Here the velocity field components are:

$$u = U + \frac{q_s}{2\pi}\frac{x}{x^2+y^2} \quad \text{and} \quad v = \frac{q_s}{2\pi}\frac{y}{x^2+y^2} \tag{7.40}$$

The stagnation point ($\mathbf{u} = 0$) is located at $x = -a = -q_s/2\pi U$, and $y = 0$, and the value of the stream function on the stagnation streamline ($\theta = \pi$, $r = a$) is $\psi = q_s/2$.



The streamlines that emerge vertically from the stagnation point form a semi-infinite body with a smooth nose, generally called a *half-body*. These stagnation streamlines divide the field into regions external and internal to the half body. The internal flow consists entirely of fluid emanating from the source, and the external region contains fluid from upstream of the source. The half-width of the body, $h$, can be found from (7.39) with $\psi = q_s/2$:

$$h = q_s(\pi - \theta)/2\pi U \tag{7.41}$$

Far downstream ($\theta \to 0$), the half-width tends to $h_{\max} = q_s/2U$.

The pressure distribution can be found from the Bernoulli equation:

$$\frac{1}{2}\rho|\mathbf{u}|^2 + p = p_\infty + \rho U^2/2 \quad \longrightarrow \quad C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2} = 1 - \frac{|\mathbf{u}|^2}{U^2} \tag{7.42}$$

Here, $C_p$ is the Euler number or pressure coefficient, a dimensionless excess pressure.

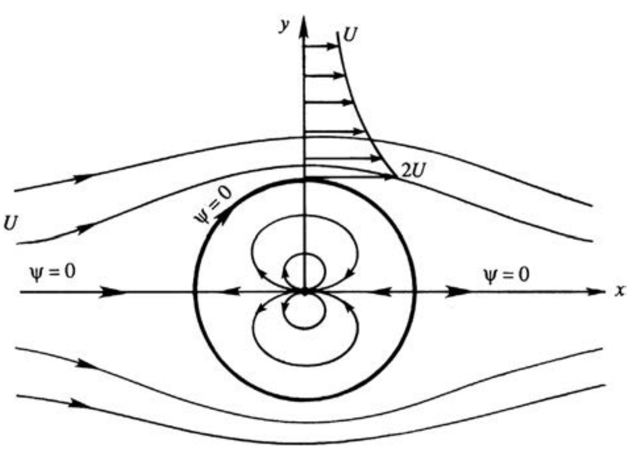
### 7.4.2 Uniform Flow + Doublet

A second example of flow construction via superposition is a horizontal free stream $U$ and a doublet with strength $\mathbf{d} = -2\pi U a^2 \mathbf{e}_x$:

$$\phi = Ux + \frac{Ua^2 x}{x^2 + y^2} = U\left(r + \frac{a^2}{r}\right)\cos\theta, \quad \text{or} \quad \psi = Uy - \frac{Ua^2 y}{x^2 + y^2} = U\left(r - \frac{a^2}{r}\right)\sin\theta \tag{7.43}$$

The velocity field is:

$$u_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta, \text{ and } u_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta \tag{7.44}$$




Here, $\psi = 0$ at $r = a$ for all values of $\theta$, showing that the streamline $\psi = 0$ represents a circular cylinder of radius $a$. The velocity components on the surface of the cylinder are $u_r = 0$ and $u_\theta = -2U\sin\theta$, so the cylinder-surface pressure coefficient is:

$$C_p(r = a, \theta) = 1 - 4\sin^2\theta \tag{7.45}$$

This flow has two stagnation points at at $r - \theta$ coordinates, $(a, 0)$ and $(a, \pi)$. The cylinder-surface pressure minima occur at $r - \theta$ coordinates $(a, \pm\pi/2)$ where the surface flow speed is maximum.

**d'Alembert's Paradox.** The symmetry of the pressure distribution implies that there is no net pressure force on the cylinder. In fact, a general result of the two-dimensional ideal flow theory is that a steadily moving body experiences no drag. This result is at variance with observations and is sometimes referred to as *d'Alembert's paradox*. In real flows, there are two differences with ideal flow related to drag:

1. In real flows tangential viscous stresses develop on the solid surface, commonly known as skin friction, causing *viscous drag* forces.
2. However, most of the drag often comes from flow separation and the formation of a wake. When a wake is present, the flow loses 'fore-aft' symmetry and the surface pressure on the downstream side of the object is smaller than that predicted by ideal flow theory, resulting in *pressure drag*.

### 7.4.3 Uniform Flow + Doublet + Irrotational Vortex

Although there is no net drag force on a circular cylinder in steady irrotational flow, there may be a lateral or lift force perpendicular to the free stream when circulation is added. Consider the flow field (7.43) with the addition of a point vortex of circulation $-\Gamma$ at the origin that induces a clockwise velocity:

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta + \frac{\Gamma}{2\pi}\ln\left(\frac{r}{a}\right) \tag{7.46}$$

The tangential velocity component at any point in the flow is:

$$u_\theta = -\frac{\partial\psi}{\partial r} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi r} \tag{7.47}$$

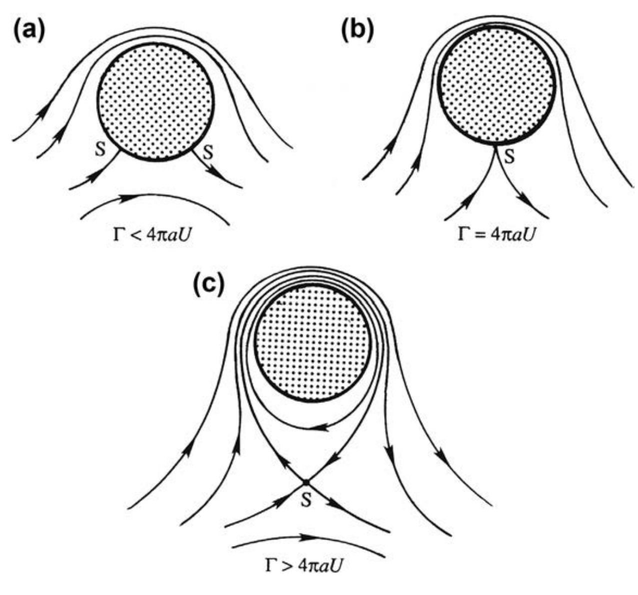
At the surface of the cylinder ($r = a$), the fluid velocity is entirely tangential and is given by

$$u_\theta(r = a, \theta) = -2U\sin\theta - \Gamma/2\pi a \tag{7.48}$$

which vanishes if:

$$\sin\theta = -\Gamma/4\pi aU \tag{7.49}$$

For $\Gamma < 4\pi aU$, two values of $\theta$ satisfy (7.49), implying that there are two stagnation points on the cylinder's surface. The stagnation points progressively move down as $\Gamma$ increases and coalesce when $\Gamma = 4\pi aU$. For $\Gamma > 4\pi aU$, the stagnation point moves out into the flow along the negative $y$-axis.




The cylinder surface pressure is found by substituting the velocities at the surface of the cylinder into the Bernoulli equations, which gives:

$$p(r = a, \theta) = p_\infty + \frac{1}{2}\rho\left[U^2 - \left(-2U\sin\theta - \frac{\Gamma}{2\pi a}\right)^2\right] \tag{7.50}$$

The upstream-downstream symmetry of the flow implies that the pressure force on the cylinder has no stream-wise component (d'Alembert's paradox). The lateral pressure force (per unit length perpendicular to the flow plane) is

$$L = -\int_0^{2\pi} p(r = a, \theta)\mathbf{n}dl \cdot \mathbf{e}_y = -\int_0^{2\pi} p(r = a, \theta) \sin\theta a d\theta, \tag{7.51}$$

where $\mathbf{n} = \mathbf{e}_r$ is the outward normal from the cylinder, and $dl = ad\theta$ is a surface element of the cylinder's cross-section; $L$ is known as the lift force in aerodynamics. Evaluating the integral using the previously found pressure produces the *Kutta-Zhukhovsky lift theorem*:

$$L = \rho U \Gamma \tag{7.52}$$

This result is valid for irrotational flow around any two-dimensional object; not just for circular cross-sections. The result that the upward lift force $L$ is proportional to the clockwise circulation $\Gamma$ is of fundamental importance in aerodynamics. It turns out that the magnitude of circulation around an airfoil (cross-section of an aeroplane wing) in a flow depends on the flow speed $U$, and the shape and orientation of the airfoil.

# Chapter 9. Laminar Flow

## 9.1 Introduction

For low values of the Reynolds number, the entire flow may be influenced by viscosity, and the inviscid flow theory is no longer even approximately correct.
Viscous flows generically fall into two categories, *laminar* and *turbulent*, but the boundary between them is imperfectly defined. At low flow rates, fluids move in parallel layers (laminae) with no unsteady macroscopic mixing or overturning motion of the layers. Such smooth orderly flow is called laminar. However, if the flow rate was increased beyond a certain critical value, fluids show the presence of unsteady, apparently chaotic three-dimensional macroscopic mixing motions. Such irregular disorderly flow is called turbulent.
The transition from laminar to turbulent flow always occurs at or near a fixed value of the Reynolds number, $\mathrm{Re} = Ud/\nu \sim 2000$ to 3000 where $U$ is the velocity averaged over the tube's cross-section, $d$ is the tube diameter, and $\nu = \mu/\rho$ is the kinematic viscosity.

Starting point for laminar flows in which viscous effects are important throughout the flow are the constant-density, constant-viscosity Navier Stokes momentum equations

$$D\mathbf{u}/Dt = -(1/\rho)\nabla p + \nu\nabla^2\mathbf{u} \tag{9.1}$$

and the continuity equation

$$\nabla \cdot \mathbf{u} = 0 \tag{9.2}$$

The velocity boundary conditions on a solid surface are:

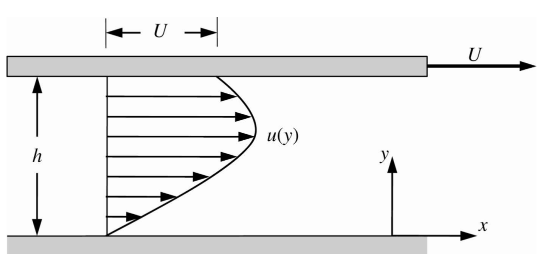
$$\mathbf{n} \cdot \mathbf{u}_s = (\mathbf{n} \cdot \mathbf{u})_{\text{on the surface}} \;\text{ and }\; \mathbf{t} \cdot \mathbf{u}_s = (\mathbf{t} \cdot \mathbf{u})_{\text{on the surface}}\,, \tag{9.3}$$

where $\mathbf{u}_s$ is the velocity of the surface, $\mathbf{n}$ is the normal to the surface, and $\mathbf{t}$ is the tangent to the surface in the plane of interest. Here fluid density will be assumed constant, and the frame of reference will be inertial. Thus, gravity can be dropped from the momentum equation as long as no free surface is present. However, when free surfaces are present, density will vary near the surface and the gravitational body force should reappear in the Navier-Stokes equation.

## 9.2 Exact Solutions for Steady Incompressible Viscous Flow

Because of the presence of the nonlinear acceleration term $\mathbf{u} \cdot \nabla\mathbf{u}$ in the total derivative in (9.1), very few exact solutions of the Navier-Stokes equations are known in closed form. An example of an exact solution is for steady laminar flow between infinite parallel plates. Within the entrance length, the derivative $\partial u/\partial x$ is not zero so the continuity equation $\partial u/\partial x + \partial v/\partial y = 0$ requires that $v \neq 0$, so that the flow is not parallel to the walls within the entrance length. Such a flow is said to be fully developed when its velocity profile $u(x, y)$ becomes independent of the downstream coordinate $x$ so that $u = u(y)$ alone and $v = 0$.

### 9.2.1 Steady Flow between Parallel Plates




The flow is sustained by an externally applied pressure gradient ($\partial p/\partial x \neq 0$) in the $x$-direction, and horizontal motion of the upper plate at speed $U$ in the $x$-direction. For a fully developed flow, $\mathbf{u} = (u(y), 0, 0)$, and the

$x$- and $y$-momentum equations reduce to:

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{d^2 u}{dy^2}, \quad \text{and} \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial y}. \tag{9.4}$$

From this, the velocity profile can be found to be:

$$u(y) = \frac{U}{h}y - \frac{1}{2\mu}\frac{dp}{dx}y(h-y) \tag{9.5}$$

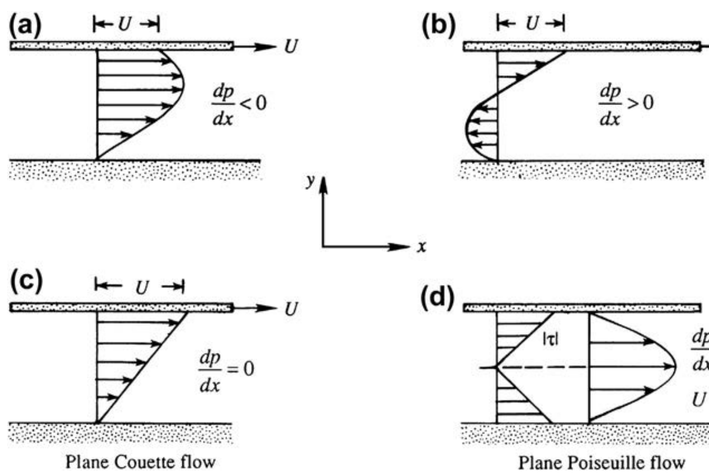
The volume flow rate $q$ per unit width of the channel (out of plane) is:

$$q = \int_0^h u dy = U\frac{h}{2}\left[1 - \frac{h^2}{6\mu U}\frac{dp}{dx}\right] \tag{9.6}$$

so that the average velocity is:

$$V \equiv \frac{q}{h} = \frac{1}{h}\int_0^h u dy = \frac{U}{2}\left[1 - \frac{h^2}{6\mu U}\frac{dp}{dx}\right] \tag{9.7}$$

Depending on the configuration of $U$ and $(\partial p/\partial x$, there are various possible cases.




**Plane Couette flow.** When the flow is driven by the motion of the upper plate alone, without any externally imposed pressure gradient, it is called a *plane Couette flow*. Velocity profile:

$$u(y) = \frac{U}{h}y - \cancel{\frac{1}{2\mu}\frac{dp}{dx}y(h-y)} \tag{9.8}$$

Shear stress:

$$\tau = \mu(du/dy) = \mu U/h \tag{9.9}$$

**Plane Poiseuille flow.** When the flow is driven by an externally imposed pressure gradient without motion of either plate, it is called a *plane Poiseuille flow*. Velocity profile:

$$u(y) = \cancel{\frac{U}{h}y} - \frac{1}{2\mu}\frac{dp}{dx}y(h-y) \tag{9.10}$$

Shear stress:

$$\tau = \mu\frac{du}{dy} = -\left(\frac{h}{2} - y\right)\frac{dp}{dx}, \tag{9.11}$$

### 9.2.2 Steady Flow in a Round Tube

A second geometry for which there is an exact solution is steady, fully developed laminar flow through a round tube of constant radius $a$, frequently called *circular Poiseuille flow*.
The flow is axisymmetric, and cylindrical coordinates with $z$ along the tube will be used. When the flow is fully developed, the only non-zero component of velocity is the axial velocity $u_z(R)$, and $\mathbf{u} = (0, 0, u_z(R))$ automatically satisfies the continuity equation. The radial and angular equations of motion reduce to:

$$0 = \partial p/\partial\varphi \text{ and } 0 = \partial p/\partial R, \tag{9.12}$$

so $p$ is a function of $z$ alone. The $z$-momentum equation gives:

$$0 = -\frac{dp}{dz} + \frac{\mu}{R}\frac{d}{dR}\left(R\frac{du_z}{dR}\right) \tag{9.13}$$

The velocity distribution follows as:

$$u_z(R) = \frac{R^2 - a^2}{4\mu}\frac{dp}{dz} \tag{9.14}$$

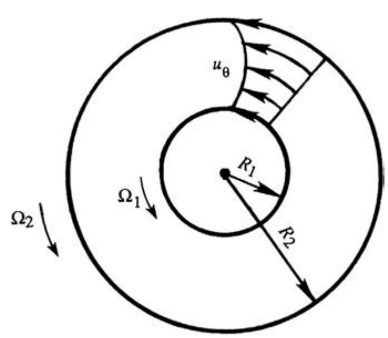
and the shear stress (and maximum shear stress) as:

$$\sigma_{zR} = \tau = \mu\frac{\partial u_z}{\partial R} = \frac{R}{2}\frac{dp}{dz'} \quad \text{and} \quad \tau_0 = \frac{a}{2}\frac{dp}{dz} \tag{9.15}$$

Integrating the velocity distribution over the cross-sectional area yields the volumetric flow rate $Q$ and average velocity over the cross-section $V$ as

$$Q = \int_0^a u(R)2\pi R dR = -\frac{\pi a^4}{8\mu}\frac{dp}{dz'}, \quad \text{and} \quad V = \frac{Q}{\pi a^2} = -\frac{a^2}{8\mu}\frac{dp}{dz} \tag{9.16}$$

### 9.2.3 Steady Flow between Concentric Rotating Cylinders




A third example is the steady flow between two concentric, rotating cylinders, also known as *circular Couette flow*. The flow is axisymmetric, and cylindrical coordinates with $z$ directed into the plane are used. When the flow is fully developed, the only non-zero component of velocity is the angular $u_\phi(R)$, and $\mathbf{u} = (0, u_\phi(R), 0)$ automatically satisfies the continuity equation. The momentum equations for the radial and tangential directions are:

$$-\frac{u_\varphi^2}{R} = -\frac{1}{\rho}\frac{dp}{dR}, \quad \text{and} \quad 0 = \mu\frac{d}{dR}\left[\frac{1}{R}\frac{d}{dR}(Ru_\varphi)\right] \tag{9.17}$$

The velocity distribution is given by

$$u_\varphi(R) = \frac{1}{R_2^2 - R_1^2}\left\{\left[\Omega_2 R_2^2 - \Omega_1 R_1^2\right]R - \left[\Omega_2 - \Omega_1\right]\frac{R_1^2 R_2^2}{R}\right\} \tag{9.18}$$

**Limiting case 1.** As $R_2 \to \infty$ with $\Omega_2 = 0$, the velocity distribution reduces to

$$u_\varphi(R) = \frac{\Omega_1 R_1^2}{R} \tag{9.19}$$

which is identical to that of an ideal vortex, see (5.2), for $R > R_1$

$$u_\theta = \Gamma/2\pi R \tag{9.20}$$

when $\Gamma = 2\pi\Omega_1 R_1^2$. This is the only example of a viscous solution that is completely irrotational.

**Limiting case 2.** As $R_1 \to \infty$ with $\Omega_1 = 0$, the velocity distribution reduces to

$$u_\varphi(R) = \Omega_2 R \tag{9.21}$$

This produces steady viscous flow within a cylindrical tank of radius $R_2$ rotating at rate $\Omega_2$, which is the velocity field of rigid body rotation ($S_{\text{ij}} = 0$).

## 9.6 Low Reynolds Number Viscous Flow Past a Sphere

Consider the problem of steady constant-density flow of a viscous fluid at speed $U$ around an object (size $L$), governed by the steady form of equation (9.1):

$$\rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mu \nabla^2 \mathbf{u} \tag{9.22}$$

The focus will be on the limit of low Reynolds numbers. In that case, inertia is negligible and the pressure term is scaled using a viscous stress

$$\mu \partial u / \partial y \sim \mu U / L \tag{9.23}$$

(instead of the dynamic (inertial) pressure $\rho U^2$ used in ch. 4.11) leading to

$$p^* = (p - p_\infty)\, L / \mu U \tag{9.24}$$

Such that:

$$\nabla p = \mu \nabla^2 \mathbf{u} \tag{9.25}$$

This equation is valid for *creeping flows* (Re $\to 0$), such as falling mist droplets in the atmosphere, or the flow of molten plastic during moulding.

We begin by considering the near-field flow around a stationary sphere of radius a placed in a uniform stream of speed $U$ with Re $\to 0$. The problem is axisymmetric (analyzed in spherical coordinates), that is, the flow patterns are identical in all planes parallel to $U$ and passing through the center of the sphere. The velocity components can be written as:

$$u_r = U \cos\theta \left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right), \quad \text{and} \quad u_\theta = -U \sin\theta \left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right) \tag{9.26}$$

The pressure is found by integrating the momentum equation $\nabla p = \mu \nabla^2 \mathbf{u}$. The result is:

$$p - p_\infty = -\frac{3\mu a U \cos\theta}{2r^2} \tag{9.27}$$

The maximum $p - p_\infty = 3\mu U / 2a$ occurs at the forward stagnation point ($\theta = \pi$), while the minimum $p - p_\infty = -3\mu U / 2a$ occurs at the rear stagnation point ($\theta = 0$).

The drag force D on the sphere can be determined by integrating its surface pressure and shear stress distributions to find:

$$D = 6\pi \mu a U \tag{9.28}$$

which is known as *Stokes' law of resistance.*

The drag coefficient, $C_D$, defined by (4.47) with $A = \pi a^2$, for Stokes' sphere is:

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 \pi a^2} = \frac{24}{\text{Re}} \tag{9.29}$$

**Millikan's experiment.** Millikan (1911) used Stokes' drag formula to measure the charge of an electron (and won the Nobel prize).

*Experiment*: Two horizontal parallel plates are charged by a battery. Oil is sprayed through a very fine hole in the upper plate and develops static charge (+) by losing a few ($n$) electrons in passing through the small hole. If the plates are charged by the switch (upper plate negative and bottom plate positive), then an upward electric force $neE$ will act on each of the drops, where $E = V_b/L$ is the electric field, and $e$ is the elementary charge of an electron.

*Step 1*: Find the radius $a$ of the falling droplets at zero field $E$.
The falling body reaches its terminal velocity $U$ when it no longer accelerates, at which point the effective weight (weight minus buoyancy force) equals the viscous drag :

$$(4/3)\pi a^3 g\,(\rho' - \rho) = 6\pi\mu a U$$

with $\rho'$ and $\rho$ the density of the droplet and air, $\mu$ the viscosity of air and $U$ is measured by the observation telescope. This equation can be solved for $a$.

*Step 2*: Find the charge $ne$ of the droplets.
Next, the upper plate is negatively charged, so that the electric field is pointing upward. The droplets will reverse direction and accelerate upward. Once the terminal velocity $U_\mathrm{u}$ is reached, there must be a balance between upward forces (electric + buoyancy) and downward forces (weight + drag):

$$6\pi\mu U_u a + (4/3)\pi a^3 g\,(\rho' - \rho) = neE,$$

which, upon measurement of $U_\mathrm{u}$ and knowledge of $a$ from step 1 , can be solved for $ne$. As $n$ must be an integer, data from many droplets may be compared to identify the minimum difference that must be $e$, the charge of a single electron.